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Enumeration of Independent Sets of Graphs

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Abstract. In this paper, enumeration of Independent Sets of Graphs is NP-hard, our ways are combinatorial counting methods. In the use of the number $N(G,k)$ of $S^{(n)}$-factors with exactly k components, the authors gain the representing formula of the number $\alpha(G)$ of all k independent sets of graphs and the equality $\alpha(G) = A(\bar{G})$, where $A(\bar{G})$ is the number of all $S^{(n)}$-factors in $\bar{G}$, and present the explicit formulas of enumeration of independent sets of graphs for a great deal of graphs. Finally, applications for the mean color numbers $\mu(G)$ are given.

1. Introduction

In this paper, the authors discuss enumeration of the number $\alpha(G)$ of all k independent sets of graphs by means of counting theory of $S^{(n)}$-factors. Enumeration of Independent Sets of Graphs is NP-hard.

Definition 1.1. *For $S^{(n)} = \{K_i : 1 \le i \le n\}$, $n \ge 1$, K_i is a complete graph with i vertices, if M is a subgraph of any graph G, and each component of M is all isomorphic to some element of $S^{(n)} = \{K_i : 1 \le i \le n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of G, then M is called one $S^{(n)}$-factor of G.*

Let $N(G,k)$ denote the number of $S^{(n)}$-factors with exactly k components. $A(G)$ is the number of all $S^{(n)}$-factors, namely, $A(G) = \sum_{k=1}^{n} N(G,k)$.

Definition 1.2. *For any n-coloring Γ of G, let $L(\Gamma)$ denote the actual number of colors used, the average of $L(\Gamma)^{,s}$ over all n-coloring Γ is called the mean color number. (see [2])*

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Let $\mu(G)$ denote the mean color number of any graph G. In the paper[3], F. M. Dong gained bounds for mean color numbers of graphs.

In the paper[4], Yang has given the recurrence relation of $A(G)$. In the paper[5], Yang derived the recurrence formula of regular m-furcating tree. So far, we have solved counting problems of $N(G,k)$(see[6]), involving the representing formula of $N(G,k)$and counting formulas of a great deal of graphs, for examples, any path, cycle, complete graph, $O \bigodot C_n$, wind graph K_n^d, complete d-partite graph, n-2-regular graph and n-3-regular graph. In this paper, the authors present the formulas of classes of graphs $\alpha(G)$ by means of counting theory of $N(G,k)$. Specially, $\alpha(G)$ of any tree is given. Finally, applications for $\mu(G)$ of any tree are given.

2. Lemmas

Here we will denote that $\alpha(G,k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G. $\alpha(G)$ is the number of all partitions of $V(G)$ into non-empty independent sets of any graph G, namely, $\alpha(G) = \sum_{k=1}^{n} \alpha(G,k)$.

Lemma 2.1 ([12])**.** *Suppose $N(G,k)$ is the number of $S^{(n)}$-factors with exactly k components in G, and the chromatic polynomial of graph G is $f(G,t) = \sum\limits_{p=1}^{n} Y_p t^p$, then the representing formula of $\alpha(G,k)$ is the following*

$$\alpha(G,k) = \sum_{p=k}^{n} N(K_p,k) Y_p \ ,$$

where

$$N(K_p,k) = \sum_{\sum_{i=1}^{p} ib_i = p \, \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i\geq 2}^{p} \frac{1}{b_i!(i!)^{b_i}}.$$

Lemma 2.2 ([12])**.** *There exists the equality $\alpha(G,k) = N(\bar{G},k)$.*

Lemma 2.3 ([12])**.** *Suppose $\mu(G)$ is the mean colour number of G, then $\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k N(\bar{G},k)}{\sum\limits_{k=1}^{n} (n)_k N(\bar{G},k)}$, where $N(G,k)$ is the number of all $S^{(n)}$-factors with exactly k components in G.*

Lemma 2.4. *If $S(n,k)$ is the Stirling number of the second kind, then $N(K_n,k) = S(n,k)$, where K_n is a complete graph with n vertices.*

Lemma 2.5 ([9])**.** *Let Bell number $B(n) = \sum_{k=1}^{n} S(n,k)$. Then $B(n) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!}$*

Lemma 2.6 ([5]). *If $G \cap H = \phi$ for any graphs G and H, then* $N(G \cup H, k) = \sum\limits_{l+m=k} N(G,l)N(H,m)$.

3. Main Theorems

Theorem 3.1. *If $\alpha(G)$ is the number of all partitions of $V(G)$ into non-empty independent sets of any graph, then $\alpha(G) = \sum\limits_{k=1}^{n}\sum\limits_{p=k}^{n} N(K_p,k)Y_p$, where Y_p are coefficients of the chromatic polynomial of $f(G,t)$.*

Proof. Because of $\alpha(G) = \sum\limits_{k=1}^{n} \alpha(G,k)$, and by Lemma 2. 1

$$\alpha(G,k) = \sum_{p=k}^{n} N(K_p,k)Y_p \ ,$$

where Y_p are coefficients of the chromatic polynomial of $f(G,t)$. Then $\alpha(G) = \sum\limits_{k=1}^{n}\sum\limits_{p=k}^{n} N(K_p,k)Y_p$, where Y_p are coefficients of the chromatic polynomial of $f(G,t)$. □

Theorem 3.2. *There exists the equality $\alpha(G) = A(\bar{G})$, where $A(\bar{G})$ is the complementary graph $\bar{G}$ of G.*

Proof. With $\alpha(G) = \sum_{k=1}^{n} \alpha(G,k)$, and by Lemma 2. 2 $\alpha(G,k) = N(\bar{G},k)$, so we gain $\alpha(G) = \sum_{k=1}^{n} N(\bar{G},k) = A(\bar{G})$. □

4. Classes of Graphs $\alpha(G)$

In this section, we will obtain classes of graphs $\alpha(G)$, for examples, any (n-2)-regular graph, (n-3)-regular graph and complete d-partite graph, tree.

Theorem 4.1. *If G is a (n-2)-regular graph with n (even 2m) vertices, then $\alpha(G) = 2^m$.*

Proof. Let G be a (n-2)-regular graph with n (even 2m), then $\bar{G}$ is a 1-regular graph, namely, $\bar{G} = K_2 \bigcup K_2 \bigcup \cdots \bigcup K_2$, and the number of K_2 is m. (see [7]) By Corollary 4. 1 we have

$$N(\bar{G},k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \binom{\frac{n}{2}}{k-\frac{n}{2}}, & \frac{n}{2} \le k \le n\,. \end{cases}$$

Finally, $\alpha(G) = \sum_{k=1}^{n} N(\bar{G},k) = \sum\limits_{k=m}^{2m} \binom{m}{k-m} = \sum\limits_{p=0}^{m} \binom{m}{p} = 2^m$. □

Theorem 4.2. *If G is a (n-3)-regular graph with n vertices, $n \geq 6$ and $\bar{G} \cong C_n$, then $\alpha(G) = L_n$, where L_n is Lucas number.*

Proof. Let G be a (n-3)-regular graph with n vertices, $n \geq 6$ and $\bar{G} \cong C_n$, then we have (see [7])

$$N(\bar{G}, k) = N(C_n, k) = \begin{cases} 0\,, & 1 \leq k < \frac{n}{2}\,, \\ \frac{n}{k}\begin{pmatrix} k \\ n-k \end{pmatrix}, & \frac{n}{2} \leq k \leq n\,. \end{cases}$$

By Corollary 4. 1 $\alpha(G, k) = N(\bar{G}, k)$, then the result is given the following $\alpha(G) = \sum\limits_{k=\frac{n}{2}}^{n} \frac{n}{k}\begin{pmatrix} k \\ n-k \end{pmatrix}$ (n even), $\alpha(G) = \sum\limits_{k=[\frac{n}{2}]+1}^{n} \frac{n}{k}\begin{pmatrix} k \\ n-k \end{pmatrix}$ (n odd). When $n \in N$, $\alpha(G) = L_n$, where L_n is Lucas number, $a = \frac{1-\sqrt{5}}{2}, b = \frac{1+\sqrt{5}}{2}$. Also L_n is the number of all $S^{(n)}$-factors, $L_n = a^n + b^n$, $a = \frac{1-\sqrt{5}}{2}, b = \frac{1+\sqrt{5}}{2}$ (see [4]). □

Corollary 4.3. *If G is a (n-3)-regular graph with n vertices, and*

$$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

$n_1 + n_2 + \cdots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, $n \geq 6$, the number of $n_j = 3$ is l, then $\alpha(G) = 5^l \prod_{j=1}^{q-l} L_{n_j}$, where L_{n_j} is the n_j-th Lucas number, and $\sum_{j=1}^{q-l} n_j = n - 3l$, when $n_j = 3$,

$$N(C_3, l) = \begin{cases} 1\,, & l = 1\,, \\ 3\,, & l = 2\,, \\ 1\,, & l = 3\,, \end{cases}$$

when $n_j \geq 4$,

$$N(C_{n_j}, l_j) = \begin{cases} 0\,, & 1 \leq l_j < \frac{n_j}{2}\,, \\ \frac{n_j}{l_j}\begin{pmatrix} l_j \\ n_j - l_j \end{pmatrix}, & \frac{n_j}{2} \leq l_j \leq n_j\,. \end{cases}$$

Proof Because of $\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q}$, $n_1 + n_2 + \cdots + n_q = n$, and $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, $n \geq 6$, by

Lemma 2.6 then

$$\begin{aligned} N(\overline{G},k) &= N(C_{n_1}\cup C_{n_2}\cup\cdots\cup C_{n_q},k)\\ &= \sum_{l_1+l_2+\cdots+l_q=k} N(C_{n_1},l_1)N(C_{n_2},l_2)\cdots N(C_{n_q},l_q)\\ &= \sum_{l_1+l_2+\cdots+l_q=k}\prod_{j=1}^{q} N(C_{n_j},l_j). \end{aligned}$$

By Theorem 2, we have

$$\alpha(G)=A(\bar{G})=\sum_{k=1}^{n} N(\bar{G},k)=\sum_{k=1}^{n}\sum_{l_1+l_2+\cdots+l_q=k}\prod_{j=1}^{q} N(C_{n_j},l_j),$$

when $n_j=3$,

$$N(C_3,l)=\begin{cases}1\,, & l=1\,,\\ 3\,, & l=2\,,\\ 1\,, & l=3\,,\end{cases}$$

when $n_j\geq 4$,

$$N(C_{n_j},l_j)=\begin{cases}0\,, & 1\leq l_j<\frac{n_j}{2}\,,\\ \dfrac{n_j}{l_j}\dbinom{l_j}{n_j-l_j}, & \frac{n_j}{2}\leq l_j\leq n_j\,.\end{cases}$$

Finally, $\alpha(G)=\prod_{j=1}^{q}\sum_{j=1}^{n_j} N(C_{n_j},l_j)=\prod_{j=1}^{q}A(C_{n_j})=5^l\prod_{j=1}^{q-l}(a^{n_j}+b^{n_j})=5^l\prod_{j=1}^{q-l}L_{n_j}$, where L_{n_j} is the n_j-th Lucas number, and $\sum_{j=1}^{q-l}n_j=n-3l$.

Theorem 4.4. *If G is a complete d-partite graph $K_{n_1,n_2,\cdots,n_d}$, and $n_1+n_2+\cdots+n_d=n$, then $\alpha(G)=\prod_{j=1}^{d}B(n_j)$, where $B(n_j)$ is Bell number, $n_j,n\in N$.*

Proof. Because of $G=K_{n_1,n_2,\cdots,n_d}$, and $n_1+n_2+\cdots+n_d=n$, then $\bar{G}=K_{n_1}\bigcup K_{n_2}\bigcup\cdots\bigcup K_{n_d}$, $n_1+n_2+\cdots+n_d=n$, $K_{n_i}\cap K_{n_j}=\phi$ for any i and j, $i \neq j$, $3 \leq n_j < n, 1 \leq j \leq d, d \geq 2$, by Lemma 2.6 we have

$$\begin{aligned} N(\overline{G}, k) &= N(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k) \\ &= \sum_{l_1+l_2+\cdots+l_d=k} N(K_{n_1}, l_1) N(K_{n_2}, l_2) \cdots N(K_{n_d}, l_d) \\ &= \sum_{l_1+l_2+\cdots+l_d=k} \prod_{j=1}^{d} N(K_{n_j}, l_j). \end{aligned}$$

With Lemma 2.4 $N(K_n, k) = S(n, k)$, then

$$\begin{aligned} N(\overline{G}, k) &= N(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k) \\ &= \sum_{l_1+l_2+\cdots+l_d=k} N(K_{n_1}, l_1) N(K_{n_2}, l_2) \cdots N(K_{n_d}, l_d) \\ &= \sum_{l_1+l_2+\cdots+l_d=k} \prod_{j=1}^{d} S(n_j, l_j) \end{aligned}$$

By Theorem 2, then we have

$$\begin{aligned} \alpha(G) &= A(\bar{G}) = \sum_{k=1}^{n} N(\bar{G}, k) = \sum_{k=1}^{n} \sum_{l_1+l_2+\cdots+l_d=k} \prod_{j=1}^{d} S(n_j, l_j) \\ &= \prod_{j=1}^{d} \sum_{l_j=1}^{n_j} S(n_j, l_j) = \prod_{j=1}^{d} B(n_j), \end{aligned}$$

where $B(n_j)$ is Bell number $n_j, n \in N$.

□

Corollary 4.5. *If G is a complete tri-partite graph K_{n_1,n_2,n_3}, and $n_1 + n_2 + n_3 = n$, then $\alpha(G) = B(n_1)B(n_2)B(n_3)$, where $B(n_j)$ is Bell number, $n_j \in N, j = 3$.*

Proof. It is easily proved by Theorem 4.5 Here we omit the proof. □

Corollary 4.6. *If G is a complete tri-partite graph $K_{n,n,n}$, then $\alpha(G) = B^3(n)$, where $B(n)$ is Bell number, $n \in N$.*

Proof. It is easily proved by Corollary 4.2 Here we omit the proof. □

Corollary 4.7. *If G is a complete bi-partite graph $K_{n,n}$, then $\alpha(G) = B^2(n)$, where $B(n)$ is Bell number, $n \in N$.*

Proof. It is easily proved by Corollary 4.3 Here we omit the proof. □

Theorem 4.8. *If G is a tree with n vertices, then*

$$\alpha(G)=\sum_{k=1}^{n}\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k),$$

where

$$N(K_p\,,k)=\sum_{\sum\limits_{i=1}^{p} ib_i=p,\ \sum\limits_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2\leq p\ \ k\leq n\ .$$

Proof. If G is a tree with n vertices, then the chromatic polynomial of G is $f(T,t)=t(t-1)^{n-1}=\sum\limits_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}t^{k+1}$

Coefficients of the chromatic polynomial of G are $Y_p=(-1)^{n-p}\dbinom{n-1}{p-1}, 1\leq p\leq n$. By Theorem 4. 1 $\alpha(G)=\sum\limits_{k=1}^{n}\sum\limits_{p=k}^{n}N(K_p,k)Y_p$, then we have

$$\alpha(G)=\sum_{k=1}^{n}\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k),$$

where

$$N(K_p\,,k)=\sum_{\sum\limits_{i=1}^{p} ib_i=p,\ \sum\limits_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2\leq p\ \ k\leq n\ .$$

□

Corollary 4.9. *Suppose P_n is any path with length n, and has n+1 vertices, then* $\alpha(P_n)=\sum\limits_{k=1}^{n+1}\sum\limits_{p=k}^{n+1}(-1)^{n+1-p}\binom{n}{p-1}N(K_p\,,k)$*, where*

$$N(K_p\,,k)=\sum_{\sum\limits_{i=1}^{p} ib_i=p,\ \sum\limits_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2\leq p\ \ k\leq n+1\ .$$

Proof. Because P_n is a special tree with $n+1$ vertices, by Theorem 6 we derive the result $\alpha(P_n)=\sum\limits_{k=1}^{n+1}\sum\limits_{p=k}^{n+1}(-1)^{n+1-p}\binom{n}{p-1}N(K_p\,,k)$, where

$$N(K_p\,,k)=\sum_{\sum\limits_{i=1}^{p} ib_i=p,\ \sum\limits_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2\leq p\ \ k\leq n+1\ .$$

So far, we have solved NP-hard problem of enumeration of independent sets of graphs, involving the representing formula of enumeration of independent sets of graphs and explicit formulas for a great of graphs.

Applications

In the section, applications are given for $N(G,k)$ and $\alpha(G,k)$ of graphs.

Theorem 4.10. *If G is any tree with n vertices, then* $\frac{\mu(G)}{n} \sim (1-\frac{1}{e})$, $n \sim \infty$.

Proof. If G is a tree with n vertices, then we have

$$N(\bar{G},k) = \sum_{p=k}^{n} (-1)^{n-p} \binom{n-1}{p-1} S(n,k).$$

By Lemma 3.3 we have

$$\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}{\sum\limits_{k=1}^{n} (n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}.$$

On the other hand, if G is a tree with n vertices, and $\mu(G) = n(1-\frac{P(G,n-1)}{P(G,n)})$, $P(G,t) = t(t-1)^{n-1}$, then $\mu(G) = n - \frac{(n-2)^{n-1}}{(n-1)^{n-2}}$. Finally, we derive the equality

$$\begin{aligned} & \frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}{\sum\limits_{k=1}^{n} (n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)} \\ = & \; n - \frac{(n-2)^{n-1}}{(n-1)^{n-2}} \frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}{\sum\limits_{k=1}^{n} n(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)} \\ = & \; 1 - \frac{(n-2)^{n-1}}{n(n-1)^{n-2}} \end{aligned}$$

So that we have the asymptotic formula $\frac{\mu(G)}{n} \sim (1-\frac{1}{e})$, $n \sim \infty$. □

Corollary 4.11. *There exists the combinatorial formula*

$$\frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}{\sum\limits_{k=1}^{n} n(n)_k \sum\limits_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)} = 1 - \frac{(n-2)^{n-1}}{n(n-1)^{n-2}}.$$

Proof. The combinatorial formula from the proving course 0f Theorem 4.10 Omitted. □

Corollary 4.12. *There exists the asymptotic formula*

$$\frac{\sum_{k=1}^{n} k(n)_k \sum_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)}{\sum_{k=1}^{n} n(n)_k \sum_{p=k}^{n} (-1)^{n-p}\binom{n-1}{p-1} S(n,k)} \sim (1-\frac{1}{e}), n \sim \infty,$$

where $S(n,k)$ is the Stirling number of the second kind.

Proof. The asymptotic formula from the proving course of Theorem 4.10 Omitted. □

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