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Intervals for option prices *

Rituparna Sen †

Abstract

An important aspect of the stock price process, which has often been ignored in the financial literature, is that prices on organized exchanges are restricted to lie on a grid. We consider continuous-time models for the stock price process with random waiting times of jumps and discrete jump size. We consider a class of pure jump processes that are "close" to the Black-Scholes model in the sense that as the jump size goes to zero, the jump model converges to geometric Brownian motion. We study the changes in pricing caused by discretization. Upper and lower bounds on option prices are developed. We study the performance of these intervals with real data.

1 Introduction

Most of the standard literature in finance for pricing and hedging of contingent claims assumes that the underlying assets follow a geometric Brownian motion as given by the Black and Scholes (1973) model. However diffusion models are not really valid descriptions of data when it comes to microstructure. An alternative approach is to use pure-jump models. Eberlein and Jacod (1997) argue why a pure-jump process is more appropriate than a continuous one. The case for modeling asset price processes as purely discontinuous processes is also presented in a review paper by Madan (1999). The arguments address both the empirical realities of asset returns and the implications of the economic principle of no

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†*Mailing Address:* Department of Statistics, University of California at Davis, One Shields Avenue Davis, CA 95616 U.S.A. Ph: (530) 752-7623 *E–mail:* rsen@wald.ucdavis.edu *Webpage:* http://www-stat.ucdavis.edu/ rsen
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arbitrage. Some popular pure-jump models are the Binomial Model with Random Time Steps of Dengler and Jarrow (1997), Variance Gamma model of Madan et al (1998), Normal Inverse Gaussian model of Barndorff-Neilsen (1998), Hyperbolic Distribution of Eberlein and Keller (1995), the CGMY process of Carr et al (2002). These are all parametric models and there is no clear way to verify the model assumptions. We take a distribution-free approach with minimal model assumptions and compute the range of values the option price can take over all possible jump distributions that belong to a large class. A somewhat similar approach is taken by Eberlein and Jacod (1997) who consider the class of all pure-jump Levy processes. However, the bounds that they derive for option prices are too large to be practicable.

Another aspect of the price process that has often been ignored is that security prices typically move in fixed units like 1/16 or 1/100 of a dollar. This does not present any particular problem when data are observed, say daily. Current technology however permits almost continuous observation, and estimation procedures based on discretely observed diffusions would then require throwing away data so as to fit the model. Harris (1991) and Brown et al (1991) argue the economic reason for the traders as well as institutions to maintain a non-trivial tick-size. Gottlieb and Kalay (1985) and Ball (1988) examine the biases resulting from the discreteness of observed stock prices. One approach is to assume an underlying continuous model and treat observed prices as realizations with rounding. Another approach is to do away with the continuous path assumption completely and concentrate on processes that move on a grid. A very common alternative model in financial literature that does this is the Binomial model introduced by Cox et al (1979). This has been extended to discrete time multinomial lattices in Boyle (1988) and Madan et al (1989). Continuous time versions of the binomial model, as studied by Dengler and Jarrow (1997), Korn et. al. (1998), Sen (2005) attempt to integrate the randomness of jump times with the discreteness of jump sizes. We extend their approach by allowing jumps of random size, not necessarily ± 1.

As soon as we move out of the realm of continuous processes, the market becomes incomplete and the distribution of stock prices is not uniquely determined by no-arbitrage restrictions. We consider a class of jump processes that are "close" to the Black-Scholes model in the sense that as the jump size goes to zero, the jump model converges to geometric Brownian motion, which is the process for stock prices in the Black-Scholes model. We do not assume any further structure on the distributions. This requirement of convergence gives us the rate of events of the jump process and the first few moments of the jumps. Restricting to these models produces bounds on option prices that are small enough to be of practical use, without imposing further assumptions on the model. Thus we get an idea of how much difference it makes if we release the continuous path and normality assumptions of Brownian motion. We impose very few moment conditions, thereby allowing the thick tailed distributions that are observed in the empirical study of stock prices. The purpose of the paper is two-fold: first, to study the deviation of option prices from those predicted by continuous models; and second, to obtain the range of option prices when the distribution of the stock price belongs to the class of discontinuous models under consideration. We obtain interval estimates for option prices in the neighborhood of the point estimates obtained from the Black-Scholes model.

The paper is organized as follows. Section 2 describes the proposed model, explains the underlying assumptions and discusses the estimation of the parameters in the model. Section 3 compares the range of option prices under the model to the Eberlein and Jacod range and describes the algorithm to derive the upper and lower bounds on option prices under all the permissible models. In Section 4 we present summary of results from actual data to compare the various methods of computation and study the performance of the model.

2 The proposed model

We start with the simple model of jumps of size $\pm c$ and event rate proportional to the present stock price (linear jump rate). This is the discrete state-space version of the popular affine jump diffusion models, for example see Duffie et al (2000). This model is also studied by Korn et al (1998) where, assuming that the risk-neutral distribution is a linear jump process, they obtain the implied jump rates by inverting the price of a market traded option and price other options using these rates. However, we show that the linear intensity birth-death model with constant intensity rate is not adequate. In section 2.1, we describe the linear birth-death model of Korn et al (1998) and show that the stock price process under this model converges in probability as $c \to 0$ to a deterministic process. So we need to either change the event rate or introduce jumps of size greater than 1. In Sen (2005), we study the quadratic (intensity proportional to square of stock price) birth-death model with random event rate. In this paper we study general jump models with jump size greater than one and linear intensity with constant rate. In section 2.2 we introduce these general jump models. We state the precise theorem and conditions involving convergence of the jump models to geometric Brownian motion, the continuous path models for stock prices in Black-Scholes option pricing theory. This convergence result is a general technique. In fact, if the underlying security is believed to have different properties than those predicted by the Black-Scholes model in the limit, then we can similarly derive different conditions on the class of "close" jump processes. As an illustration, similar conditions for convergence to the Cox-Ingersoll-Ross model are stated. Hence under these modified conditions, we have a model for interest rates that is the discretized version of the Cox-Ingersoll-Ross model.

2.1 The linear birth-death model

Suppose that the stock price S_t is a birth and death process with jump size c, jump intensity $\lambda_t S_t/c$, and probability of a positive jump p_t for some positive parameters λ_t and p_t. Let $N_t = S_t/c$. For example, S_t is price of stock in dollars, N_t in cents, $c = 1/100$. N_t is modeled as a *non-homogeneous* (birth and death rates per individual depend on t), *linear* (rates are proportional to number of individuals present) birth and death process. We suppose that there is a risk-free interest rate ρ_t. Let $\mathrm{P}_k(t) = \mathrm{P}(N_t = k)$. The Kolmogorov's forward equations are:

$$\begin{aligned}
\mathrm{P}'_k(t) &= -k\lambda_t \mathrm{P}_k(t) + (k-1)\lambda_t p_t \mathrm{P}_{k-1}(t) + (k+1)\lambda_t(1-p_t)\mathrm{P}_{k+1}(t) \quad k \geq 1 \\
\mathrm{P}'_0(t) &= \lambda_t(1-p_t)\mathrm{P}_1(t) \quad k \geq 1
\end{aligned}$$

$$\phi(u,t) = \sum_{k=0}^{\infty} \mathrm{P}_k(t) u^k = \left[1 - \frac{1}{\frac{1}{a_t(1-u)} + b_t}\right]^{N_0} \tag{1}$$

where $a_t = \exp\{\int_0^t \lambda_s(2p_s - 1)\, ds\}$ and $b_t = \int_0^t \lambda_s p_s / a_s ds$. The derivation of $\phi(u,t)$ is given in Appendix B.

$$\mathrm{E}(N_t) = \left.\frac{\partial \phi}{\partial u}\right|_{u=1} = N_0 \left(1 - \frac{1}{\frac{1}{a_t(1-u)} + b_t}\right)^{N_0 - 1} \left.\frac{\frac{1}{a_t(1-u)^2}}{\left(\frac{1}{a_t(1-u)} + b_t\right)^2}\right|_{u=1} = N_0 a_t$$

Similarly, we can derive the variance of N_t. The distribution of N_t is the sum of N_0 iid random variables with mean a_t and variance $(a_t^{-1} + 2b_t - 1)a_t^2$. So $\mathrm{E}(S_t) = S_0 a_t$ and

$\text{Var}(S_t) = c(a_t^{-1} + 2b_t - 1)a_t^2 S_0$.

$$\text{P}(\mid S_t - e^{\int_0^t \rho_s ds} S_0 \mid > \epsilon) \leq \frac{c(a_t^{-1} + 2b_t - 1)a_t^2 S_0}{\epsilon^2}$$

When $c \longrightarrow 0$, $S_t \stackrel{\text{P}}{\longrightarrow} \exp\{\int_0^t \rho_s ds\} S_0$. Thus the simple model of jumps of size $\pm c$ and event rate proportional to the present stock price converges to a deterministic process in probability.

2.2 Introducing distribution on the size of jumps

Suppose now that for each n, the stock price $S_t^{(n)} = N_t^{(n)}/n$ where $N_t^{(n)}$ is a sequence of integer valued jump processes. That is, the grid size is $c = 1/n$ and we consider a sequence of random processes with grid size decreasing to 0. We assume initial stock price $S_0^{(n)}$ is the same for all n. The number of jumps $\xi_t^{(n)}$ is assumed to be a counting process with rate $N_t^{(n)} \sigma_t^2$ and the random jump size of $N_t^{(n)}$ is denoted by $Y_t^{(n)}$. Let $\mathcal{F}_u^{(n)} = \sigma\{N_u, 0 \leq u \leq t\}$. Under some assumptions on the conditional distribution of $Y_t^{(n)}$ that are outlined in Proposition 2.2.1, as $n \to \infty$, the sequence of random processes $S_t^{(n)}$ converge in distribution to process S_t which evolves as

$$\ln S_t = \ln S_0 + \int_0^t (\rho_u - \frac{1}{2}\sigma_u^2) du + \int_0^t \sigma_u dW_u \tag{2}$$

where W_t is standard Weiner process. This is the stochastic differential equation governing the stock price process in the Black-Scholes model of asset pricing.

To illustrate that this method is quite general, we can consider the interest rate process. Suppose the interest rate process $R_t^{(n)} = N_t^{(n)}/n$ where the process $N_t^{(n)}$ is as described above. We assume initial interest rate $R_0^{(n)}$ is the same for all n. Under some assumptions on the conditional distribution of $Y_t^{(n)}$ that are outlined in Proposition 2.2.2, as $n \to \infty$, the sequence of random processes $R_t^{(n)}$ converge in distribution to process R_t which evolves as :

$$R_t = R_0 + \int_0^t a(b - R_u) du + \int_0^t \sigma \sqrt{R}_u dW_u \tag{3}$$

where W_t is standard Weiner process. This is the stochastic differential equation governing the interest rate according to the Cox-Ingersoll and Ross model for interest rates (Ref Section 21.5 of Hull (1999))

PROPOSITION 2.2.1: *Let $N_t^{(n)}$ and $Y_t^{(n)}$ be as described above. Then the process $S_t^{(n)} = N_t^{(n)}/n$ converges in distribution to a process S_t which evolves as in (2) if:*

(B1)
$$\sup_{s\le t}\left|\ln\left(1+\frac{Y_s^{(n)}}{N_{s-}^{(n)}}\right)\right| \xrightarrow{\mathrm{P}} 0 \quad \forall t$$

(B2)
$$\int_0^T \mathrm{E}\left[\ln\left(1+\frac{Y_t^{(n)}}{N_{t-}^{(n)}}\right)\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T (\rho_t-\sigma_t^2)dt$$

(B3)
$$\int_0^T \mathrm{E}\left[\left\{\ln(1+\frac{Y_t^{(n)}}{N_{t-}^{(n)}})\right\}^2\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt$$

A set of sufficient conditions for (B2)- (B3) to hold is: $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = \rho_t/\sigma_t^2$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = N_{t-}^{(n)}$ and $\mid Y_t^{(n)} \mid \le kN_{t-}^{(n)\delta}$ where $0<k<1$ and $\delta < 2/3$.
Proof The proof is given in Appendix A.

PROPOSITION 2.2.2: *Let $N_t^{(n)}$ and $Y_t^{(n)}$ be as described above. Then the process $R_t^{(n)} = N_t^{(n)}/n$ converges in distribution to a process R_t, which evolves as in (3), if*

(C1)
$$\sup_{s\le t}\left|\sqrt{Y_s^{(n)}+N_{s-}^{(n)}}-\sqrt{N_{s-}^{(n)}}\right| \xrightarrow{\mathrm{P}} 0 \quad \forall t$$

(C2)
$$\int_0^T \left\{\mathrm{E}\left[\left(\sqrt{Y_t^{(n)}+N_{t-}^{(n)}}-\sqrt{N_{t-}^{(n)}}\right)\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\frac{\sigma_t^2}{\sqrt{n}} - \frac{a(b-N_{t-}^{(n)}/n)}{\sqrt{N_{t-}^{(n)}/n}}\right\} dt \xrightarrow{\mathrm{P}} 0$$

(C3)
$$\int_0^T \mathrm{E}\left[\left(\sqrt{Y_t^{(n)}+N_{t-}^{(n)}}-\sqrt{N_{t-}^{(n)}}\right)^2\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\frac{\sigma_t^2}{n} dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt$$

A set of sufficient conditions for (C2)- (C3) to hold is: $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = ab/(N_{t-}^{(n)}\sigma^2) - (a-\sigma^2/4)/(n\sigma^2)$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = n$ and $\mid Y_t^{(n)} \mid \le kN_{t-}^{(n)\delta}$ where $0<k<1$ and $\delta < 2/3$. According to Feller (1951), initial values can be prescribed arbitrarily for the model (3) and they uniquely determine a solution. This solution is positivity preserving and norm decreasing.
Proof The proof is similar to that of Proposition 2.2.1

For the rest of the paper, unless otherwise mentioned, we shall restrict ourselves to stock price processes S_t, that under the physical measure are pure-jump processes with the jump time ξ_t, a counting process with rate $S_t/c\lambda$ and jump size cY_t, where Y_t is an integer valued random variable with $\mathrm{E}(Y_t \mid S_t) = \nu$ and $\mathrm{E}(Y_t^2 \mid S_t) = S_t/c$. We shall denote the class of probability measures associated with such processes by $\mathcal{M}$. We have shown that if we let c to go to zero, then under some regularity conditions, such processes converge to geometric Brownian motion with drift λ/ν and volatility λ. We shall denote $\mathrm{P}(Y_t = i \mid S_t = cj)$ by $p(i,j)$.

3 Upper and lower bounds on option price

We shall assume, for this chapter, that there is a constant interest rate ρ. We restrict ourselves to the class $\mathcal{M}$ of probability measures described in the end of Section 2.2 with $\nu = \rho/\lambda$. We do not have a unique distribution for the stock price. This is because the distribution of jump size is not uniquely specified by the conditions imposed. We get a class of models each of which gives a different price for options. A similar problem is addressed in Eberlein and Jacod (1997) who consider the class of all pure jump Levy processes. They derive upper and lower bounds for option prices when the distribution of the stock price process belongs to a large class of distributions. In Section 3.1 we present the Eberlein-Jacod bounds and show that these bounds hold if the class of distributions is $\mathcal{M}$. We show that in this case, there exists a smaller upper bound than the Eberlein and Jacod upper bound. We also show that the lower bound is sharp; that is, there exist a sequence of distributions in $\mathcal{M}$, under which the option price converges to the Eberlein and Jacod lower bound. We cannot obtain any sharp upper bounds theoretically for the models under consideration. So in Section 3.2 we present an algorithm to obtain these.

3.1 Comparing to Eberlein-Jacod bounds

Suppose there is a constant interest rate ρ. Let $\gamma(Q) = \mathrm{E}_Q[e^{-\rho T} f(S_T)]$ be the expected discounted payoff of an option under the measure Q. Assume

(D) $$f \text{ is convex, and } \quad 0 \leq f(x) \leq x \quad \forall x > 0$$

LEMMA 1: *Under each $Q \in \mathcal{M}$, $e^{-\rho t} S_t$ is a martingale.*

Proof The proof is given in Appendix B
It is shown in Eberlein and Jacod (1997), that under reasonable conditions on Q, and f satisfying (D), the following holds:

$$e^{-\rho T} f(e^{\rho T} S_0) < \gamma(Q) < S_0 \tag{4}$$

PROPOSITION 3.1.1: *The Eberlein-Jacod bounds 4 hold for $Q \in \mathcal{M}$ and f satisfying (D).*

Proof Since f is convex, by Lemma 1, under each $Q \in \mathcal{M}$ the process $A_t = f(e^{\rho(T-t)} S_t)$ is a Q-submartingale. So $\gamma(Q) = e^{-\rho T} \mathrm{E}_Q[A_T] \geq e^{-\rho T} f(e^{\rho T} S_0)$. We have $e^{-\rho T} f(S_T) < e^{-\rho T} S_T$ by assumption (D). So $\gamma(Q) < \mathrm{E}_Q[e^{-\rho T} S_T] = S_0$

PROPOSITION 3.1.2: *There exists a smaller upper bound for $\gamma(Q)$ than that given in 4 when $Q \in \mathcal{M}$.*

Proof Let ξ_t be the counting process of the number of jumps in the stock price.

$$\begin{aligned} e^{\rho T} \gamma(Q) &= \mathrm{E}_Q[f(S_T)] \\ &= \mathrm{E}_Q[f(S_T) I_{\{\xi_T = 0\}}] + \mathrm{E}_Q[f(S_T) I_{\{\xi_T > 0\}}] \\ &\leq \mathrm{P}(\xi_T = 0) f(S_0) + \mathrm{E}_Q[S_T I_{\{\xi_T > 0\}}] \\ &= \mathrm{P}(\xi_T = 0) f(S_0) + \mathrm{E}_Q[S_T] - E_Q[S_T I_{\{\xi_T = 0\}}] \\ &= \mathrm{P}(\xi_T = 0) f(S_0) + e^{\rho T} S_0 - S_0 \mathrm{P}(\xi_T = 0) \\ &= e^{\rho T} S_0 - (S_0 - f(S_0)) \exp\{-\frac{S_0}{c} \lambda_0 T\} \end{aligned}$$

$S_0 - \gamma(Q) = e^{-\rho T}(S_0 - f(S_0)) \exp\{-\frac{S_0}{c} \lambda_0 T\} > 0$

PROPOSITION 3.1.3: *Assuming $\rho = 0$, there exist a sequence of distributions $Q^{(m)} \in \mathcal{M}$ such that $\gamma(Q^{(m)})$ converges to the lower bound in (4) as $m \to \infty$*

Proof Define the measure $Q^{(m)}$ as follows: For each value j of ξ_T and each m, define the Markov chain $S_k^{(m,j)}$ for $1 \leq k \leq j$ by

$$S_k^{(m,j)} = \begin{cases} S_{k-1}^{(m,j)} - \sqrt{S_{k-1}^{(m,j)}} \frac{\sqrt{c}}{\sqrt{m-1}} & w.p. \;\; 1 - \frac{1}{m} \\ S_{k-1}^{(m,j)} + \sqrt{S_{k-1}^{(m,j)}} \sqrt{c} \sqrt{m-1} & w.p. \;\; \frac{1}{m} \end{cases}$$

$\mathrm{E}[S_k^{(m,j)} - S_{k-1}^{(m,j)} \mid S_{k-1}^{(m,j)}] = 0$
$\mathrm{E}[(S_k^{(m,j)} - S_{k-1}^{(m,j)})^2 \mid S_{k-1}^{(m,j)}] = c S_{k-1}^{(m,j)}$
Hence $Q^{(m)} \in \mathcal{M}$

CLAIM 3.1.1: *Given* δ*and* ϵ*, for each* j*, we can get* m_j *such that for all* $m > m_j$,

$$\mathrm{P}(|S_T^{(m,j)} - S_0| > \delta \mid \xi_T = j) < \epsilon$$

CLAIM 3.1.2: *There exists* J *such that* $\mathrm{P}(\xi_T > J) < \epsilon$

Let $n = max_{j=1}^{J} m_j$. Then $\forall m > n, \forall j < J \quad \mathrm{P}(|S_T^{(m,j)} - S_0| > \delta \mid \xi_T = j) < \epsilon$

$$\begin{aligned} \mathrm{P}(|S_T^{(m)} - S_0| > \delta) &= \sum_{j=1}^{J} \mathrm{P}(|S_T^{(m)} - S_0| > \delta \mid \xi_T = j)\mathrm{P}(\xi_T = j) \\ &\quad + \mathrm{P}(|S_T^{(m)} - S_0| > \delta \mid \xi_T > J)\mathrm{P}(\xi_T > J) \\ &\leq \epsilon \times 1 + 1 \times \epsilon \\ &= 2\epsilon \end{aligned}$$

Hence $S_T^{(m)} - S_0 \xrightarrow{\mathrm{P}} 0$. $S_T^{(m)}$ is non-negative and $\mathrm{E}(S_T^{(m)}) = \mathrm{E}(S_0)$. So $\{S_T^{(m)}\}$ is uniformly integrable. This and assumption D implies $\{f(S_T^{(m)})\}$ is uniformly integrable. $S_T^{(m)} \xrightarrow{\mathrm{P}} S_0$ and f is continuous. So $f(S_T^{(m)}) \xrightarrow{\mathrm{P}} f(S_0)$. This and uniform integrability of $f(S_T^{(m)})$ implies $\mathrm{E}(f(S_T^{(m)})) \longrightarrow \mathrm{E}(f(S_0)) = f(S_0)$
Proof of **Claim 3.1.1**

$$\begin{aligned} &\mathrm{P}(| S_T^{(m,\xi_T)} - S_0^{(m,\xi_T)} | \leq \xi_T \sqrt{S_0^{(m,\xi_T)}} \sqrt{c} \frac{1}{\sqrt{m-1}} \mid \xi_T = j) \\ \geq\ & \mathrm{P}(S_j^{(m,\xi_T)} - S_{j-1}^{(m,\xi_T)} = -\sqrt{S_{j-1}^{(m,\xi_T)}} \frac{\sqrt{c}}{\sqrt{m-1}} \\ & \&\ldots \ \& S_1^{(m,\xi_T)} - S_0^{(m,\xi_T)} = -\sqrt{S_0^{(m,\xi_T)}} \frac{\sqrt{c}}{\sqrt{m-1}}) \\ =\ & (1 - \frac{1}{m})^j \qquad \square \end{aligned}$$

Proof of **Claim 3.1.2**

$$\mathrm{P}(\xi_T > J) \leq \frac{1}{J}\mathrm{E}(\xi_T) = \frac{1}{J}\mathrm{E}\int S_t \frac{\lambda}{c} dt = \frac{S_0 \lambda t}{cJ} < \epsilon \text{ for } J \text{ sufficiently large}\square$$

When $\rho \neq 0$, we need to let the grid size go to 0 to obtain a sequence of measures that converge to the lower bound.

PROPOSITION 3.1.4: *When* $\rho \neq 0$*, there exist a sequence of distributions* $Q^{(c,m)} \in \mathcal{M}$ *such that* $\gamma(Q^{(c,m)})$ *converges to* $e^{-\rho T} f(S_0(1 + \rho T))$ *as* $c \to 0$ *and* $m \to \infty$

Proof The proof of Proposition 3.1.3 can be extended to nonzero interest rate with the following modifications: For each grid size c, define the Markov chain $S_k^{(m,\xi_T)}$ for $1 \le k \le \xi_T$ by

$$S_k^{(m,\xi_T)} = \begin{cases} S_{k-1}^{(m,\xi_T)} + \frac{c\rho}{\lambda} - \sqrt{S_{k-1}^{(m,\xi_T)} - \frac{c\rho^2}{\lambda^2}}\frac{\sqrt{c}}{\sqrt{m-1}} & w.p. \;\; 1-\frac{1}{m} \\ S_{k-1}^{(m,\xi_T)} + \frac{c\rho t}{\lambda} + \sqrt{S_{k-1}^{(m,\xi_T)} - \frac{c\rho^2}{\lambda^2}}\sqrt{c}\sqrt{m-1} & w.p. \;\; \frac{1}{m} \end{cases}$$

Given $\xi_T = j$ and all jumps are negative,

$$\begin{aligned} |S_T - S_0 - \frac{jc\rho}{\lambda}| &\le \frac{\sqrt{c}}{\sqrt{m-1}} \sum_{i=1}^{j} \sqrt{S_i - \frac{c\rho^2}{\lambda^2}} \\ &\le \frac{\sqrt{c}}{\sqrt{m-1}} \xi_T \sqrt{S_0 + \frac{jc\rho}{\lambda} - \frac{c\rho^2}{\lambda^2}} \overset{m\to\infty}{\longrightarrow} 0 \end{aligned}$$

So given δ, ϵ, j, can find m large enough so that $\mathrm{P}(|S_T^{(m,j)} - S_0 - \xi_T c\rho/\lambda| > \delta/2 \mid \xi_T = j) < \epsilon$. $\mathrm{P}(|S_T - S_0 - S_0\rho t| > \delta \mid \xi_T = j) = \mathrm{P}(|S_T - S_0 - \xi_T c\rho/\lambda| > \delta/2 \mid \xi_T = j) + \mathrm{P}(|\xi_T c\rho/\lambda - S_0\rho t| > \delta \mid \xi_T = j) < \epsilon + \epsilon$. This is because $c\xi_T \overset{\mathrm{P}}{\longrightarrow} \lambda S_0 T$ as $c \to 0$ since $d < \xi >_t = S_t\lambda/c$ and $d < \xi, \xi >_t = S_t\lambda/c$. The rest of the proof is same as the case $\rho = 0$ except that instead of S_0 we now have $S_0 + S_0\rho t$. $\mathrm{E}(f(S_T^m)) \longrightarrow \mathrm{E}(f(S_0 + S_0\rho t)) = f(S_0 + S_0\rho t)$. □
If ρt is small, $S_0 + S_0\rho t$ is approximately $S_0 e^{\rho t}$.
$\gamma(Q_{m,c,\rho}) = \mathrm{E}(e^{-\rho t} f(S_T^{(m,c,\rho)})) \longrightarrow e^{-\rho t} f(S_0 e^{\rho t})$ as $m \to \infty, c \to 0, \rho \to 0$

3.2 Obtaining the upper and lower bounds

In Section 3.2.1 we describe a dynamic programming algorithm to get the maximum price of an option with payoff $f(S_T)$ when the distribution of the stock price process S_t belongs to the class $\mathcal{M}$. The same algorithm with the maximum at the intermediate steps replaced by minimum will give the minimum price. This procedure gives a range for possible option prices when the stock price process has a distribution $Q \in \mathcal{M}$. Let ξ_t=number of jumps in the stock price till time t and let $N_t = S_t/c$. The frequency distribution of ξ_T is obtained in Section 3.2.2.

3.2.1 The Algorithm

- For each m such that $\mathrm{P}(\xi_T = m) > \epsilon$,
- For each $i \le m$, going down over the integers

- For each value k of $N_{T_{i-1}}$
- maximize $\mathrm{E}(f_i(l+Y_i)|\xi_T = m, N_{T_{i-1}} = k)$ over the distribution on Y_i where

$$
\begin{aligned}
f_i(x) &= (x - \frac{K}{c})_+ && \text{for } i = m \\
f_i(x) &= \max \mathrm{E}(f(x+Y_i)|\xi_T = m, N_{T_{i-1}} = x) && \text{for } 1 \le i \le m-1
\end{aligned}
$$

The maximum value is $f_1(N_0)$. The problem reduces to maximizing $\mathrm{E}(f_i(l+Y_i)|\xi_T = m, N_{T_{i-1}} = k)$ over the distribution on Y_i. Let $p_{y,k} = \mathrm{P}(Y_i = y|N_{T_{i-1}} = k)$. We have the constraints: $\sum p_{y,k} = 1, \sum y p_{y,k} = \rho/\lambda, \sum y^2 p_{y,k} = k$

$$
\mathrm{E}(f_i(l+Y_i)|\xi_T = m, N_{T_{i-1}} = k) = \frac{\sum_y f_i(l+Y_i) p_{y,k} \mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y)}{\sum_y p_{y,k} \mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y)}
$$

$$
\mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y) = \int_0^T q_{i,k,y}(t) Q_{m-i,k+y}(T-t)dt
$$

where $Q_{m,k}(t) = \mathrm{P}(\xi_t = m|N_0 = k)$ is given by Claim 3.2.1 and $q_{i,k,y}(t)$ is the conditional density of T_i given $N_{T_{i-1}} = k, Y_i = y$. To obtain $q_{i,k,y}(t)$, observe that $T_i = T_{i-1} + \Delta T_i$. The conditional distribution of ΔT_i is $\mathrm{Exp}(N_{i-1}\sigma^2)$. T_{i-1} is independent of $N_{T_{i-1}} = k, Y_i = y$. The unconditional distribution is: $\mathrm{P}(T_{i-1} \le t) = \mathrm{P}(\xi_t \ge i-1) = 1 - \sum_{j=0}^{l-2} Q_{j,N_0}(t)$

We have to maximize the ratio of 2 linear functions of p_y under three linear constraints. That is: max $x'y/x'z$ under three linear constraints on x. Suppose at the maximum $x'z = \mu$. Then at that point $x'y$ is maximized subject to 4 linear constraints. This will be a 4-point distribution. So the maximizing p_y is supported on 4 points. Let the four points be y_1, y_2, y_3, y_4. From the three constraints, we can express $p_{y_1}, p_{y_2}, p_{y_3}$ as linear functions of p_{y_4}. Then we have to maximize a ratio of 2 linear functions of p_{y_4}. This is a monotone function of p_{y_4}. Hence the maximum occurs at a boundary. So we actually have a three point distribution where the maximum is attained. The algorithm is to check through all the three point distributions of Y_i and check where the maximum occurs.

An alternative procedure here is to do linear programming. But it was found that both linear programming and checking through all possible three point distributions took comparable amount of computational time. In fact we can characterize and eliminate a lot of 3-point combinations from the search list since all p_is need to be positive and not all combinations satisfy this. On the other hand, since linear programming gives a numerical maximum, the result has the same minimum value numerically but is not in general supported on three points and is therefore difficult to interpret. Hence our simulations were all carried out by checking through all admissible three point distributions.

If the number of possible values of the stock price is n, then the number of possible jump combinations that we need to check naively is n^3. However, this number is greatly reduced since all these combinations cannot support probability distributions with the given constraints. Suppose the jump distribution is supported on three points $y_1 < y_2 < y_3$. Want to find $(p_{y_1,k}, p_{y_2,k}, p_{y_3,k})$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{pmatrix} \begin{pmatrix} p_{y_1,k} \\ p_{y_2,k} \\ p_{y_3,k} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\rho}{\lambda} \\ k \end{pmatrix}$$

Solving this, we get:

$$\begin{aligned} p_{y_1,k} &= \tfrac{\lambda y_3 y_2 - \rho y_3 - \rho y_2 + \lambda k}{\lambda(y_3-y_1)(y_2-y_1)} \\ p_{y_2,k} &= \tfrac{-\lambda y_1 y_3 + \rho y_1 + \rho y_3 - \lambda k}{\lambda(y_2-y_1)(y_3-y_2)} \\ p_{y_3,k} &= \tfrac{y_1 y_2 \lambda - \rho y_1 - \rho y_2 + k\lambda}{\lambda(y_3-y_1)(y_3-y_2)} \end{aligned}$$

Must have: $(1)(\rho/\lambda)^2 < k \quad (2) y_1 < \rho/\lambda < y_3$
Fix y_1. $p_{y_2,k}, p_{y_3,k} > 0 \Rightarrow y_2 < (k\lambda - \rho y_1)/(\rho - \lambda y_1) < y_3$
Fix $y_2 < \rho/\lambda$. $p_{y_1,k} > 0 \Rightarrow y_3 < (k\lambda - \rho y_2)/(\rho - \lambda y_2)$
Fix $y_2 > \rho/\lambda$. $p_{y_1,k} > 0 \Rightarrow y_3 > (\rho y_2 - \lambda k)/(y_2\lambda - \rho)$

Another issue here is that for computational purposes the search for maximum needs to be restricted to finite limits. For stock prices the lower limit is always zero, since the stock price cannot be negative. Theoretically there is no upper limit. So we derive in Appendix C.1 the probability of the stock price lying below some given bounds. Then, given a probability p close to 1, we obtain the corresponding upper bound on the stock price and carry out the computation by restricting the stock price to be below that bound. This gives bounds on the option prices that hold with probability p.

3.2.2 Distribution of ξ_T

$$\text{Let} \quad \begin{aligned} P_{n,k}(t) &= \mathrm{P}(N_t = n | N_0 = k) \\ Q_{m,k}(t) &= \mathrm{P}(\xi_t = m | N_0 = k) \\ P_{n,m,k}(t) &= \mathrm{P}(N_t = n | \xi_t = m, N_0 = k) \end{aligned}$$

CLAIM 3.2.1: $Q_{m,k}(t) = \frac{(\frac{k\lambda}{\rho}+m-1)!}{(\frac{k\lambda}{\rho}-1)!} e^{-k\lambda t} \frac{(1-e^{-\rho t})^m}{m!}$

Proof For $m \geq 1$,

$$\begin{aligned} Q_{m,k}(t+dt) &= Q_{m-1,k}(t) \sum_n P_{n,m-1,k}(t) n\lambda dt + Q_{m,k}(t)(1 - \sum_n P_{n,m,k}(t) n\lambda dt) \\ &= Q_{m-1,k}(t) \mathrm{E}[N_t | \xi_t = m-1, N_0 = k] \lambda dt \\ &\quad + Q_{m,k}(t)(1 - \mathrm{E}[N_t | \xi_t = m, N_0 = k] \lambda dt) \\ &= Q_{m-1,k}(t)(k + \frac{(m-1)\rho}{\lambda}) \lambda dt + Q_{m,k}(t)\{1 - (k + \frac{m\rho}{\lambda}) \lambda dt\} \end{aligned}$$

$$Q'_{m,k}(t) = Q_{m-1,k}(t)(k + \frac{(m-1)\rho}{\lambda})\lambda - Q_{m,k}(t)(k + \frac{m\rho}{\lambda})\lambda \quad Q_{m,k}(0) = 0$$

$$Q_{0,k}(t+dt) = Q_{0,k}(t)(1 - \sum_n \mathrm{P}_{n,0}(t) n\lambda dt) = Q_{0,k}(t)\{1 - k\lambda dt\}$$

$$Q'_{0,k}(t) = -Q_{0,k}(t) k\lambda \quad Q_{0,k}(0) = 1$$

It can be easily verified that the proposed expression for $Q_{m,k}(t)$ satisfies the conditions derived above.□
For computational purposes, we have to restrict to finite values of ξ_T. It is shown in Appendix C.2 that for every c, the number of jumps in finite time is finite almost surely.

3.3 Maximum value the stock price can attain

While searching for the maximum option price, for computational purposes, we have to restrict the stock price to an interval. The lower limit of this interval is zero, because the stock price cannot be negative. There is, however, no natural upper limit. So we do a simulation study to find out how the bounds on option prices behave as we increase the limit on the maximum value of the stock price. Let us denote by *lim* this maximum value relative to the present stock price. That is, we shall consider $\{\sup_{0\leq t\leq T} S_t \leq lim \times S_0\}$.

With $S_0 = \$15$, tick size =\$1/8 and strike $K = \$12$, Table 4.2 presents $\mathrm{P}(\xi_T = m)$ and max $\mathrm{E}[(S_T - K)_+|\xi_T = m]$ for various values of m and lim. Table 4.2 does the same for the minimum. Recall that ξ_t is the number of jumps in time 0 to t. It is shown theoretically in Appendix C.1 that probability of $lim > 20$ is 0.05. It is seen here computationally that there is very little difference in price of the option between lim=20 and lim=1.5. Also, for each ξ_T we can compute the minimum and maximum for increasing values of lim and stop when there is very little increase.

If lim is too small, no possible three-tuple exists between 0 and $lim \times N_0$ so that the jump probabilities are non-negative. So some of the values are unavailable. Once there are possible three-tuples, increasing lim further doesn't change the minimum and maximum option price almost at all.

3.4 Robustness

The condition of convergence to Black -Scholes introduces constraints on the moments of the jump distribution. It might be of interest to studying how much the results are affected if we relax these constraints slightly. The first moment condition is necessary for martingale properties. So we keep it unaltered. We relax the second moment conditions by 5% and compute the maximum under these new conditions. What this means is instead of considering distributions that have second moment exactly equal to N_t, we consider those with second moment between $0.95N_t$ and $1.05N_t$. We do the same thing at 10% and 20%.

In Figure 1 we present the ratio of the maximum under the relaxed set of constraints to the maximum under the original constraints for various strike values (K) when the present stock price is \$100 and time to maturity is 3 months. It is observed that in-the-money CALL options are very robust. Also for all the options considered, the changes are almost same for 10% and 20% which suggests that the changes stabilize after sometime.

4 Real data applications

4.1 Description

Data on stock price and option price was obtained from the option-metrics database for three stocks: Ford, IBM and ABMD. The stock data is transaction by transaction. The format of the raw stock data is: SYMBOL, DATE, TIME, PRICE, SIZE, G127, CORR, COND, EX. After filtering for after hour and international market trading, the data is on tradings in NASDAQ regular hours. The prices are divided by the tick size to obtain integers.

The option data is daily best bid and ask prices. We preprocess the data to remove volume zero and symbols not equal to F,IBM or ABMD. For example, we do not consider the roots XFO, YFY, FOD and YOD which are on Ford stocks after a merger which pays \$20 per stock + 1 stock of the new company. The option data has 'date, Call/Put, expiration, best-bid, best-ask, strike'.

We shall use the data for the first day of the month as training sample and for the rest of the days as test sample. Table 4.2 summarizes the average number of trades per day for each of the stocks and the dates for the training and test samples. We estimate the risk-neutral parameters by inverting option prices in the training sample and use these estimates to predict prices of options in the test sample. The 3 stocks provide some variety. The Ford stock is a little old when the tick size used to be \$1/16 while the others have tick size \$1/100. This should shed some light on how much effect the change in tick size has on the analysis. The ABMD data is much more thinly traded than the other two, as will be evident from the plots of paths of stock prices. So while the IBM data can be well approximated by a continuous path and it might still be alright for the Ford data, the continuity assumption for the ABMD data is definitely too far-fetched.

4.2 Intervals for Option prices

For the general jump model, we obtain the intervals for option prices from the model under various values of the intensity rate parameter. For the learning sample of ABMD, we have 12 options that are traded on Feb 3, 2003. For values of the intensity parameter between 1×10^{-9} to 10×10^{-9}, we obtain the intervals based on simulations of size 1000. Figure 2 plots the length of the predicted interval against the distance of the predicted interval from the observed interval, as obtained for various values of the parameter. The range of option prices is 0-160. The average observed bid-ask spread is 25. The unit is 1c. The distance is measured as the average distance between the midpoint of bid and ask and the point on the predicted interval that is closest to the observed interval. Similar plots for the IBM and Ford data are in figures 4 (a) and 5 (a). Table 4.2 summarizes the various characteristics of the training samples for the 3 stocks. All numbers are in multiples of tick. The number of replications for the simulations are always 1000.

We choose one of the parameter values and use that to predict the range of option prices for the test samples. The various characteristics of the test samples for the 3 stocks are summarized in table 4.2. These include the average length of the predicted interval and the distance of the predicted interval from the observed interval, as obtained from the analysis. We present the plots showing the midpoints of observed bid-ask intervals and the corresponding predicted intervals for the test samples in figures 3, 4 (b) and 5 (b). For the ABMD data we also present a similar plot for the training sample in figure 3 (a).

A Proof of Proposition 2.2.1

Let X be a process driven by the stochastic differential equation $dX_t = (\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t$, where W_t is standard Weiner process. In particular X is a process with independent increments and characteristics $(\int_0^T (\rho_t - \sigma_t^2/2)dt, \int_0^T \sigma_t^2 dt, 0)$.
Let $X_t^{(n)} = \log(N_t^{(n)}/N_0^{(n)})$. From (6.10) in Mykland (1994)

$$< X_t^{(n)}, X_t^{(n)} >_t = \int_0^T \mathrm{E}\left[(\Delta X_t^{(n)})^2 \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt \tag{5}$$

$$< X_t^{(n)} >_t = \int_0^T \mathrm{E}\left[(\Delta X_t^{(n)}) \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T (\rho_t - \sigma_t^2/2) dt \tag{6}$$

(5), (6), assumption B1 and Theorem VIII.3.6 of Jacod and Shiryaev (2002) imply $X^{(n)} \xrightarrow{\mathcal{L}} X$. Since exp is a continuous function, $S^{(n)} = \exp(X^{(n)}) \xrightarrow{\mathcal{L}} \exp(X) =: S$. By Ito's formula,

$$dS_t = S_t[(\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t] + \frac{1}{2} S_t \sigma_t^2 dt = S_t \rho_t dt + S_t \sigma_t dW_t$$

We shall now prove that if $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = \rho_t/\sigma_t^2$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = N_{t-}^{(n)}$ and $\mid Y_t^{(n)} \mid \leq kN_{t-}^{(n)\delta}$ where $0 < k < 1$ and $\delta < 2/3$, then

$$\mathrm{E}\left[(\Delta X_t^{(n)}) \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 \xrightarrow{\mathrm{P}} \rho_t - \sigma_t^2$$

$$\mathrm{E}\left[(\Delta X_t^{(n)})^2 \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 \xrightarrow{\mathrm{P}} \sigma_t^2 du$$

$$\mathrm{E}\left[(\Delta X_t^{(n)}) \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 = \mathrm{E}\left[\ln\left(1 + \frac{Y_t}{N_{t-}}\right) N_{t-} \sigma_t^2 \middle| \mathcal{F}_{t-}\right]$$

$$
\begin{aligned}
\left|\ln\left(1+\frac{Y_t}{N_{t-}}\right)-\left[\frac{Y_t}{N_{t-}}-\frac{1}{2}\frac{Y_t^2}{N_{t-}^2}\right]\right|N_{t-}\sigma_t^2 &\leq \left|\sum_{j\geq 3}\frac{1}{j}(-1)^{j-1}\frac{Y_t^j}{N_{t-}^j}\right|N_{t-}\sigma_t^2\\
&\leq \sum_{j\geq 3}\frac{1}{j}\frac{\mid Y_t^j\mid}{N_{t-}^{j-1}}\sigma_t^2\\
&\leq \sigma_t^2\sum_{j\geq 3}\frac{k^j}{j}\frac{N_{t-}^{j\delta}}{N_{t-}^{j-1}}
\end{aligned}
$$

Since $j \geq 3$ and $\delta < 2/3$, $j-1-j\delta > 0$. Also, $N_{t-} \geq 1$. Hence the last expression is bounded above by $\sigma_t^2 \sum_{j\geq 3} \frac{k^j}{j}$ and goes to 0 a.s. as $n \to \infty$. Hence,

$$
\mathrm{E}\left[\ln\left(1+\frac{Y_t}{N_{t-}}\right)N_{t-}\sigma_t^2\middle|\mathcal{F}_{t-}\right] \longrightarrow \mathrm{E}\left[\left(\frac{Y_t}{N_{t-}}-\frac{1}{2}\frac{Y_t^2}{N_{t-}^2}\right)N_{t-}\sigma_t^2\middle|\mathcal{F}_{t-}\right] = \rho_t - \sigma_t^2/2
$$

The other convergence can be proved similarly by considering the Taylor expansion of $\ln(1+\frac{Y_t}{N_{t-}})^2$

B

B.1 Proof of Lemma 1

From (6.10) of Mykland (1994)

$$
d < S >_t = E(\Delta S_t \mid \mathcal{F}_{t-})N_t\lambda dt = c\nu N_t\lambda dt = \rho S_t dt
$$

Hence, $M_t = S_t - \int_0^t \rho S_u du$ is a martingale. By Ito's formula,

$$
d(e^{-\rho t}S_t) = e^{-\rho t}(dS_t - \rho S_t dt)
$$

$e^{-\rho t}S_t = \int_0^t e^{-\rho u}dM_u$ is a martingale. □

B.2 Derivation of $\phi(u,t)$ in Equation (1)

The Kolmogorov's forward equations are:

$$
\begin{aligned}
\mathrm{P}_k'(t) &= -k\lambda_t\mathrm{P}_k(t) + (k-1)\lambda_t p_t\mathrm{P}_{k-1}(t) + (k+1)\lambda_t(1-p_t)\mathrm{P}_{k+1}(t) \quad k \geq 1\\
\mathrm{P}_0'(t) &= \lambda_t(1-p_t)\mathrm{P}_1(t) \quad k \geq 1
\end{aligned}
$$

Multiplying both sides by u^k and taking sum over k, we get

$$\frac{\partial \phi}{\partial t} - (-\lambda_t u + \lambda_t p_t u^2 + (\lambda_t(1-p_t))\frac{\partial \phi}{\partial u} = 0$$

The corresponding ordinary characteristic differential equation is:

$$\frac{du}{dt} = -(-\lambda_t u + \lambda_t p_t u^2 + (\lambda_t(1-p_t))$$

Substituting $v = u - 1$ and rearranging terms, we get:

$$dv + (2p_t - 1)\lambda_t v dt = -\lambda_t p_t v^2 dt$$

Let $a_t = \int_0^t exp\{(2p_s - 1)\lambda_s\}ds$. We can do separation of variables as:

$$\frac{d(va_t)}{(va_t)^2} + \frac{\lambda_t p_t}{a_t}dt = 0$$

Let $b_t = \int_0^t \lambda_s p_s / a_s ds$. The general solution of the characteristic differential equation in implicit form is, therefore, given by:

$$C_1 = b_t - \frac{1}{va_t}$$

where C_1 is an arbitrary constant. Thus the genral solution of $\phi(u,t)$ has the structure:

$$\phi(u,t) = f\left(b_t + \frac{1}{(1-u)a_t}\right)$$

where f is a continuously differentiable function. f can be determined by making use of the initial condition $\phi(u,0) = u^{N_0}$. Hence $f(x) = 1 - 1/x$. So

$$\phi(u,t) = \left(1 - \frac{1}{\frac{1}{(1-u)a_t} + b_t}\right)^{N_0}$$

where $a_t = \int_0^t \exp\{(2p_s - 1)\lambda_s\}ds$ and $b_t = \int_0^t \lambda_s p_s / a_s ds$.

C

C.1 Probability Bounds on Stock price process

Since the discounted stock price is a martingale and interest rate is positive, the stock price process is a sub-martingale. Also, if we consider the closure of the state space on the infinite line and set the transition function as in the proof of Thm VI.2.2 of Doob (1953), then by Thm II.2.4′ of Doob, there is a standard extension of the process which is separable relative to the closed sets. Now, Theorem 3.2 of Doob Section VII.11 is applicable and we get:

$$\forall\ \epsilon > 0, \quad \epsilon \mathrm{P}\{\underset{\{o\leq t\leq T\}}{L.U.B.} S_t(\omega) \geq \epsilon\} \leq \mathrm{E}(S_T) = e^{\rho T} S_0$$

C.2 Nonexplosion of number of jumps

In the notation of Kerstind and Klebaner (1995), on the S_t scale,

$$\begin{aligned} m(z) &= \mathrm{E}(\frac{Y_n}{n}) = \frac{\rho}{n\sigma_2} \\ \lambda(z) &= nz\sigma^2 \\ \int_0^\infty \frac{1}{m(z)\lambda(z)} dz &= \int_0^\infty \frac{1}{\rho z} dz = \infty \end{aligned}$$

By Thm 1 of Kersting and Klebaner (1995)$\sum_{n=0}^{\infty}(\lambda(Z_n))^{-1} = \infty$ a. s. This is the necassary and sufficient condition for nonexplosion, see for example Chung (1967), that is there are only finitely many jumps in finite time intervals.

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TABLE I

ξ_T	prob	1	1.1	1.2	1.5	2	10	20
0	0.855970	3	3	3	3	3	3	3
1	0.132394	–	4.1603	4.1603	4.1603	4.1603	4.1603	4.1603
2	0.011074	–	–	5.3207	5.3207	5.3207	5.3207	
3	0.000663	–	–	–	6.5527	6.5526	6.5526	
4	0.000032	–	–	–	7.8014	7.8014	7.8014	
E ()	1000133	–	–	–	3.182224	3.182224	3.182224	

TABLE II

ξ_T	prob	1	1.1	1.2	1.5	2	10	20
0	0.855970	3	3	3	3	3	3	3
1	0.132394	–	4.1917	4.1917	4.1917	4.1917	4.1917	4.1917
2	0.011074	–	–	5.3522	5.3522	5.3522	5.3522	
3	0.000663	–	–	–	6.5841	6.5841	6.5841	
4	0.000032	–	–	–	7.8636	7.8947	8.3698	
E ()	1.000133	–	–	–	3.186753	3.186754	3.186769	

TABLE III

Stock	Trades per day	Dates for sample	
		Training	Test
F	1675	Dec 4, 2002	Dec 5-Dec 31, 2002
ABMD	400	Feb 3, 2003	Feb 4-Feb 28, 2003
IBM	4270	June 3, 2002	June 4-June 30, 2002

TABLE IV

Stock	Sample size	Range of option prices	Range of parameters	Average observed bid-ask spread
ABMD	12	0-160	1×10^{-9} to 10×10^{-9}	25
F	34	0-180	0.5×10^{-7} to 4.5×10^{-7}	2
IBM	92	0-3700	0.7×10^{-9} to 2×10^{-9}	10

TABLE V

Stock	Sample size	Number of replications	Parameter	Range of option prices	Average length	Average distance
ABMD	33	1000	3×10^{-9}	0-550	47.18	18.59
ABMD	33	10000	3×10^{-9}	0-550	66.56	13.06
F	578	1000	2×10^{-7}	0-256	17.14	4.33
IBM	1823	1000	1.4×10^{-9}	0-8000	273.17	55.68

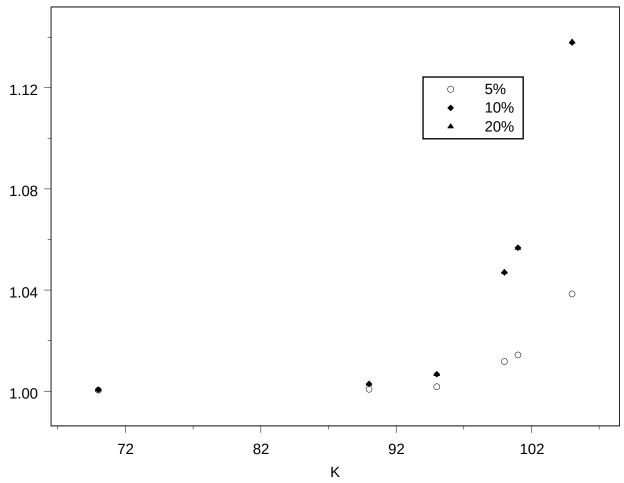

Figure 1: Plot for Robustness Study

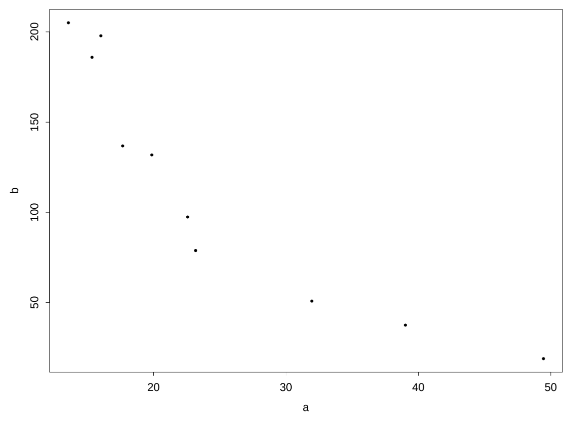

Figure 2: ABMD Training: Length of predicted interval vs distance of predicted interval from observed interval

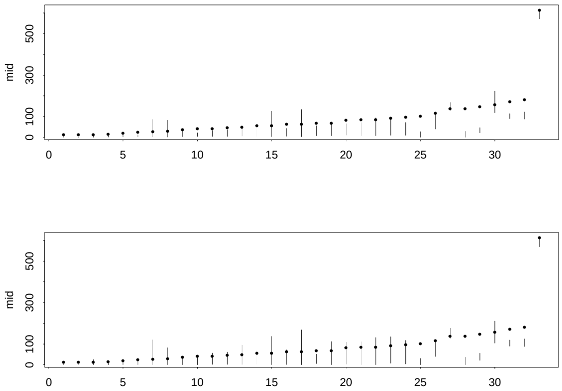

Figure 3: ABMD Intervals and bid-ask midpoint (a) Training data (b)Test data

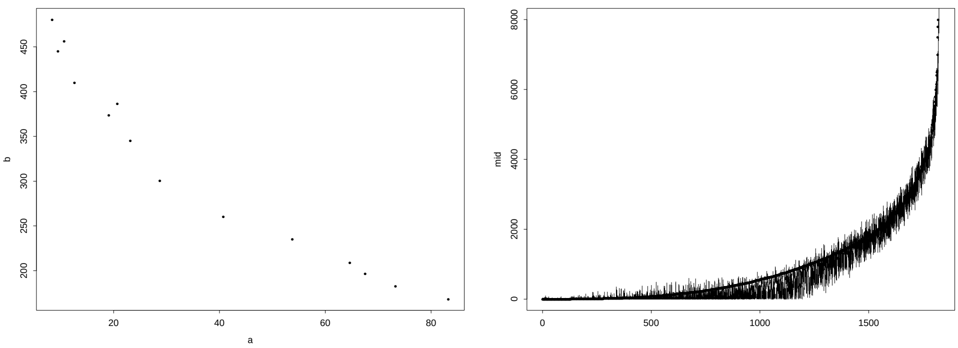

Figure 4: (a) IBM Training: Length of predicted interval vs distance of predicted interval from observed interval (b)IBM Prediction: Predicted intervals and bid-ask midpoint

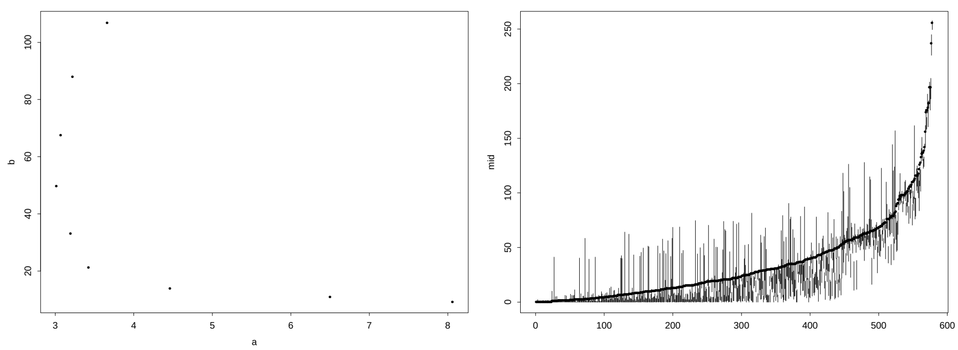

Figure 5: (a) Ford Training: Length of predicted interval vs distance of predicted interval from observed interval (b)Ford Prediction: Predicted intervals and bid-ask midpoint